

A Note on Value Sets of Polynomials over Finite Fields

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Abstract

Most results on the value sets V_f of polynomials $f \in \mathbb{F}_q[x]$ relate the cardinality $|V_f|$ to the degree of f . In particular, the structure of the spectrum of the class of polynomials of a fixed degree d is rather well known.

We consider a class $\mathcal{F}_{q,n}$ of polynomials, which we obtain by modifying linear permutations at n points. The study of the spectrum of $\mathcal{F}_{q,n}$ enables us to obtain a simple description of polynomials $F \in \mathcal{F}_{q,n}$ with prescribed V_F , especially those avoiding a given set, like cosets of subgroups of the multiplicative group \mathbb{F}_q^* . The value set count for such F can also be determined. This yields polynomials with evenly distributed values, which have small maximum count.

Keywords

Value set of a polynomial, spectrum, value set count, maximum count, permutation polynomial.

Mathematical Subject Classification

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1 Introduction

Let \mathbb{F}_q be the finite field with $q = p^r$ elements, where p is a prime, $r \geq 1$. The *value set* of a polynomial $f \in \mathbb{F}_q[x]$ is the set $V_f = \{f(c) : c \in \mathbb{F}_q\}$. Given a class of polynomials \mathcal{C} , the set $v(\mathcal{C}) = \{|V_f| : f \in \mathcal{C}\}$ is called the *spectrum*

of \mathcal{C} . Throughout we assume that polynomials are reduced $\pmod{x^q - x}$, i.e. have degree $\leq q - 1$.

Most previous results on value sets are concerned with the class $\mathcal{C}_{q,d}$ of polynomials over \mathbb{F}_q of degree d . When $d \leq 4$, the complete spectrum $v(\mathcal{C}_{q,d})$ is known, see for instance Section 8.3 of [13]. When $f \in \mathcal{C}_{q,d}$ is not a permutation, i.e. $|V_f| < q$, Wan showed in [18] that

$$|V_f| \leq q - \left\lceil \frac{q-1}{d} \right\rceil. \quad (1)$$

For many interesting results on $v(\mathcal{C}_{q,d})$, we refer to [13, 8.3.3] and [16, 8.2] and the references therein. For instance, it is shown in [10] that the gap described in (1) between permutations and non-permutations is not the only gap in $v(\mathcal{C}_{q,d})$.

In a recent paper [14], Mullen et al. obtained an upper bound, similar to that in (1), for non-permutation polynomials f of a fixed *index* ℓ ;

$$|V_f| \leq q - \frac{q-1}{\ell}.$$

We recall that any non-constant polynomial $f \in \mathbb{F}_q[x]$ of degree $\leq q - 1$ can be written uniquely as $f(x) = a(x^r g(x^{(q-1)/\ell})) + b$, with a monic polynomial $g(x)$, where ℓ is the index of f , see [1].

The well-known Lagrange Interpolation Formula is an explicit formula for polynomials with any given value set. On the other hand, giving a simple description of polynomials with specific value sets, and/or determining their properties have been of interest. Biró, for instance, shows in [3] that polynomials over the prime field \mathbb{F}_p taking only two non-zero values have degree $\geq 3(p-1)/4$, except for some special polynomials. The so-called *minimal value set polynomials* $f \in \mathcal{C}_{q,d}$ are those, satisfying $|V_f| = \lceil q/d \rceil$; the minimum value in $v(\mathcal{C}_{q,d})$. They are obtained by a variety of methods, see for example [4, 5, 6, 11]. Chou et al. study several classes of polynomials, the value sets of which lie in a subfield, see [7]. Cusick, in [8], determines five kinds of polynomials over \mathbb{F}_{2^n} with evenly distributed values, by using results from the theory of crosscorrelation of binary m -sequences. The distribution of the values of a polynomial is described in terms of the *value set count* and *maximum count* (the definitions are given in Section 2). It is shown for instance that if $n \equiv 0 \pmod{4}$, and $f(x) = (x^{2^{n/2}} + x)^2 / (x^2 + x)$ is in $\mathbb{F}_{2^n}[x]$, then 0 appears $2^{n/2}$ times in V_f , regarded as a multiset, while all the other elements appear twice, see [8]. For further results on polynomials with particular value sets, we refer the reader to [13, 8.3.3].

In this note we consider a class $\mathcal{F}_{q,n}$ of polynomials over \mathbb{F}_q , $q \geq 5$, which we obtain by modifying linear permutations at n points, where q is an odd prime power. As one might expect, the spectrum of $\mathcal{F}_{q,n}$ is of a very different nature, when compared with that of $\mathcal{C}_{q,d}$. We give a simple description of polynomials $F \in \mathcal{F}_{q,n}$ with small and large value sets, for example with $|V_F| = 2, 3, 4, q-3, q-2$. We determine V_F explicitly for such F . We also show how to obtain polynomials, whose values avoid prescribed sets, in particular any coset of any subgroup of the multiplicative group \mathbb{F}_q^* . We give value set counts for polynomials F and hence find polynomials with evenly distributed values, extending some results of Cusick mentioned above, to odd characteristic: See Corollary 6 (ii), (iii) and Remark 10. We note that the polynomials in $\mathcal{F}_{q,n}$ may also be permutations, giving rise to complete mappings, see for instance Theorem 5 and Remark 7.

We define $\mathcal{F}_{q,n}$, $n \geq 2$ as follows. Start with $g(x) = ax + b \in \mathbb{F}_q[x]$, $a, b \neq 0$. Let \mathbf{O}_n denote a set of n distinct elements $x_1, x_2, \dots, x_n \in \mathbb{F}_q$ with $x_1 = 0$ and $x_n = -b/a$. Consider the set $\mathcal{P}_{q,n}$ of permutations f of \mathbb{F}_q defined by,

$$f(x) = \begin{cases} g(x) & \text{if } x \notin \mathbf{O}_n, \\ g(x_{i-1}) & \text{if } x = x_i \in \mathbf{O}_n, \ 2 \leq i \leq n, \\ g(x_n) = 0 & \text{if } x = x_1 \in \mathbf{O}_n. \end{cases} \quad (2)$$

Adding the identity permutation to permutations in $\mathcal{P}_{q,n}$, we obtain polynomials with a variety of value sets. We put $\mathcal{F}_{q,n} = \{F(x) = f(x) + x : f \in \mathcal{P}_{q,n}\}$.

This particular choice of the polynomials f (and $F \in \mathcal{F}_{q,n}$) enables us to use the tools that are obtained in [2]. Indeed for any $f \in \mathcal{P}_{q,n}$, defined as in (2), one can uniquely determine a polynomial $P_n(x) = P_n(c_0, \dots, c_n; x)$, which satisfies the following properties. One has

$$f(\delta) = P_n(c_0, \dots, c_n; \delta) = (\dots((c_0\delta)^{q-2} + c_1)^{q-2} \dots + c_n)^{q-2} \quad (3)$$

for all $\delta \in \mathbb{F}_q$, where $c_0, \dots, c_n \in \mathbb{F}_q^*$. The recursively defined sequences

$$\alpha_k = c_{k-1}\alpha_{k-1} + \alpha_{k-2}, \quad \beta_k = c_{k-1}\beta_{k-1} + \beta_{k-2}, \quad k \geq 2, \quad (4)$$

with $\alpha_0 = 0, \alpha_1 = c_0, \beta_0 = 1, \beta_1 = 0$, point to the relation between f and $P_n(x)$ as $a = \alpha_n/\beta_{n+1}$, $b = \beta_n/\beta_{n+1}$. Moreover they satisfy $\alpha_{n+1} = 0$, and $\alpha_k \neq 0$ when $1 \leq k \leq n$.

It is easy to see that using the representation (3) for f , the set \mathbf{O}_n above can also be expressed as $\mathbf{O}_n = \{x_i : x_i = \frac{-\beta_i}{\alpha_i}, i = 1, \dots, n\}$. A procedure that yields $P_n(c_0, \dots, c_n; x)$ from a given polynomial $g(x)$ and the set \mathbf{O}_n is

essentially given in [9], and will be summarized in Section 4. See also Example 12, below.

Conversely, consider a permutation polynomial $P_n(x) = P_n(c_0, \dots, c_n; x) = (\dots((c_0x)^{q-2} + c_1)^{q-2} \dots + c_n)^{q-2} \in \mathbb{F}_q[x]$, and the elements $\alpha_k, \beta_k, 1 \leq k \leq n$, which are defined recursively by (4). Put $\overline{\mathbf{O}}_{\mathbf{n}} = \{x_i : x_i = \frac{-\beta_i}{\alpha_i}, i = 1, \dots, n\}$. If $|\overline{\mathbf{O}}_{\mathbf{n}}| = n$, $\alpha_k \neq 0$ for $1 \leq k \leq n$ and $\alpha_{n+1} = 0$, then there are uniquely determined polynomials $g(x)$ and $f(x)$, where $g(x) = (\alpha_n/\beta_{n+1})x + \beta_n/\beta_{n+1}$, $\mathbf{O}_{\mathbf{n}} = \overline{\mathbf{O}}_{\mathbf{n}}$, and $f(x)$ as defined in (2) that satisfies $f(\delta) = P_n(\delta)$ for every $\delta \in \mathbb{F}_q$. We refer the reader to [2] for details. Note that in the terminology of [2], the set

$$\overline{\mathbf{O}}_{\mathbf{n}+1} = \overline{\mathbf{O}}_{\mathbf{n}} \cup \{x_{n+1}\} = \{x_i : x_i = \frac{-\beta_i}{\alpha_i}, i = 1, \dots, n\} \cup \{\infty\} \subset \mathbb{F}_q \cup \{\infty\}$$

is called the set of *poles*, where x_{n+1} is the pole at infinity.

A representation similar to that in (3) is possible for any permutation P of \mathbb{F}_q , and it leads to the concept of the *Carlitz rank* of P . This notion was introduced in [2], and is particularly useful when n is small with respect to q , since it enables expressing permutations as fractional linear transformations, except at at most $n + 1$ elements of \mathbb{F}_q , see [12, 15, 17].

2 The Spectrum $v(\mathcal{F}_{q,n})$

Theorem 1. *The spectrum $v(\mathcal{F}_{q,n})$ satisfies*

$$v(\mathcal{F}_{q,n}) \subset \{2, 3, \dots, n+1, q-n, q-n+1, \dots, q-2, q\}.$$

Proof. Let $F(x) \in \mathcal{F}_{q,n}$ be arbitrary. We recall that $a, b \neq 0$,

$$F(\delta) = (a+1)\delta + b \text{ for } \delta \in \mathbb{F}_q \setminus \mathbf{O}_{\mathbf{n}}, \text{ and } F(x_i) = ax_{i-1} + x_i + b, i = 2 \dots n. \quad (5)$$

We also note that $|\mathbf{O}_{\mathbf{n}}| = n$ and $F(x_1) = F(0) = g(x_n) + x_1 = g(-b/a) = 0$, see (2).

Assuming $a = -1$, we get $F(\delta) = b = x_n$ for all $\delta \in \mathbb{F}_q \setminus \mathbf{O}_{\mathbf{n}}$. The cardinality of the image $F(\mathbf{O}_{\mathbf{n}})$ may vary between 1 and n . The element 0 is in V_F since $F(x_1) = 0$, while x_n is also in V_F and $x_n \neq 0$. Hence $2 \leq |V_F| \leq n+1$.

If $a \neq -1$, then $|F(\mathbb{F}_q \setminus \mathbf{O}_{\mathbf{n}})| = q-n$ and $|V_F| = q-n$ if $F(\mathbf{O}_{\mathbf{n}}) \subset F(\mathbb{F}_q \setminus \mathbf{O}_{\mathbf{n}})$. Therefore depending on the intersection of the two sets $F(\mathbf{O}_{\mathbf{n}})$ and $F(\mathbb{F}_q \setminus \mathbf{O}_{\mathbf{n}})$, the cardinality $|V_F|$ may range between $q-n$ and q , except that $|V_F| \neq q-1$ for any $F \in \mathcal{F}_{q,n}$.

Suppose that $|V_F| = q - 1$, so that $q - 2$ elements appear once in V_F , regarded as a multiset, one element α appears twice and one element β does not appear. But in this case $\sum_{c \in \mathbb{F}_q} F(c) = \sum_{c \in \mathbb{F}_q} f(c) + \sum_{c \in \mathbb{F}_q} c = 0$, while $\sum_{c \in \mathbb{F}_q} F(c) = \alpha + \sum_{c \in \mathbb{F}_q \setminus \{\beta\}} c = \alpha - \beta$. \square

Example 2. We remark that all the values listed in Theorem 1 may actually be attained. This example with $q = 13$, $n = 6$ shows that there are $F \in \mathcal{F}_{13,6}$, for which $|V_F|$ takes any value between 3 and $q - 2 = 11$ and also $q = 13$. In what follows $F_i(x) \in \mathcal{F}_{13,6}$ denotes a polynomial with $|V_{F_i}| = i$ for $1 \leq i \leq 11$, and $i = 13$. We first fix a linear polynomial $g_i(x)$ and the set $\mathbf{O}_6^{(i)}$ to obtain $f_i(x)$ as in (2) so that $F_i(x) = f_i(x) + x \in \mathcal{F}_{13,6}$ satisfies $|V_{F_i}| = i$. We use the notation $P_6^{(i)}(c_o, \dots, c_6; x)$ to specify the polynomial $P_6(x)$, satisfying $f_i(\delta) = P_6(\delta) = P_6^{(i)}(\delta)$ for any $\delta \in \mathbb{F}_q$. For instance the polynomial $P_6^{(13)}(7, 2, 3, 6, 10, 5, 2; x)$, corresponding to $g_{13}(x) = 5x + 3$ and $\mathbf{O}_6^{(13)} = \{0, 12, 1, 8, 6, 2\}$ yields $F_{13}(x)$, such that $|F_{13}(\mathbf{O}_6^{(13)})| = 6$, $|F_{13}(\mathbb{F}_q \setminus \mathbf{O}_6^{(13)})| = 7$ and $F_{13}(\mathbf{O}_6^{(13)}) \cap F_{13}(\mathbb{F}_q \setminus \mathbf{O}_6^{(13)}) = \emptyset$.

The polynomials $g_{11}(x) = 7x + 12$, $g_{10}(x) = 9x + 3$, $g_9(x) = x + 4$, $g_8(x) = 10x + 12$, with $\mathbf{O}_6^{(11)} = \{0, 4, 8, 11, 7, 2\}$, $\mathbf{O}_6^{(10)} = \{0, 6, 11, 1, 7, 4\}$, $\mathbf{O}_6^{(9)} = \{0, 11, 12, 4, 3, 9\}$, $\mathbf{O}_6^{(8)} = \{0, 6, 3, 12, 7, 4\}$ produce examples of $F_i(x)$ with $|V_{F_i}| = i$, $8 \leq i \leq 11$. The corresponding permutations are $P_6^{(11)}(5, 11, 1, 4, 7, 9, 3; x)$, $P_6^{(10)}(1, 2, 8, 3, 7, 6, 1; x)$, $P_6^{(9)}(1, 7, 2, 1, 11, 9, 2; x)$, $P_6^{(8)}(1, 2, 7, 8, 5, 4, 10; x)$.

The other end of the spectrum, i.e. the values 3, 4, 5, 6, 7 can be obtained by choosing $g_3(x) = 12x + 5$, $g_4(x) = 12x + 10$, $g_5(x) = 12x + 6$, $g_6(x) = 12x + 10$, $g_7(x) = 12x + 8$ and $\mathbf{O}_6^{(3)} = \{0, 8, 3, 2, 10, 5\}$, $\mathbf{O}_6^{(4)} = \{0, 1, 4, 7, 9, 10\}$, $\mathbf{O}_6^{(5)} = \{0, 3, 10, 2, 9, 6\}$, $\mathbf{O}_6^{(6)} = \{0, 8, 2, 11, 3, 10\}$, $\mathbf{O}_6^{(7)} = \{0, 4, 7, 9, 10, 8\}$. The corresponding permutations are $P_6^{(3)}(1, 8, 3, 9, 4, 10, 5; x)$, $P_6^{(4)}(4, 3, 1, 5, 12, 5, 1; x)$, $P_6^{(5)}(10, 3, 2, 9, 1, 11, 4; x)$, $P_6^{(6)}(9, 11, 2, 7, 8, 4, 2; x)$, $P_6^{(7)}(1, 3, 5, 4, 2, 11, 6; x)$.

The value $|V_F| = 2$ is attained when $\text{char}(\mathbb{F}_q) = p = n$, and the linear polynomial $g(x)$ and the set \mathbf{O}_n are chosen as in Theorem 9 (iv) below. For instance, $P_5(-1, -1/b, 2b, -2/b, 2b, -1/b; x)$ over F_{25} yields a polynomial F with $|V_F| = 2$, for any $b \in \mathbb{F}_{25}^*$.

In order to analyse the spectrum $v(\mathcal{F}_{q,n})$ in more detail we follow [8] and define the value set count and the maximum count. For $f \in \mathbb{F}_q[x]$, the value set count is defined in terms of the pre-images of elements in \mathbb{F}_q . It is the vector (v_0, v_1, \dots, v_M) , where $v_i = |\{\alpha \in \mathbb{F}_q : |f^{-1}(\alpha)| = i\}|$, and $M = \max_{\alpha \in \mathbb{F}_q} \{|f^{-1}(\alpha)|\}$ is the maximum count for V_f . When $v_k = 0$ for $1 < i \leq k \leq j < M$, and $v_{i-1} \neq 0$, $v_{j+1} \neq 0$, we write $(v_0, v_1, \dots, v_{i-1}, v_{j+1}, \dots, v_M)$.

When we wish to specify the elements in V_f with a given *multiplicity*, we use the notation $m(\beta) = |f^{-1}(\beta)|$ for $\beta \in V_f$.

3 The cases $n = 2, 3$

When $n = 2, 3$, it is easy to describe the polynomials $F \in \mathcal{F}_{q,n}$ and their value sets V_F explicitly.

Theorem 3. *The spectrum of the family $\mathcal{F}_{q,2}$ is*

$$v(\mathcal{F}_{q,2}) = \{3, q-2\}.$$

Proof. Let $F \in \mathcal{F}_{q,2}$ be arbitrary with $F(x) = f(x) + x$, $f \in \mathcal{P}_{q,2}$. It can be seen easily that F can be represented as

$$F(x) = \left(\left((c_0x)^{q-2} + c_1 \right)^{q-2} - \frac{1}{c_1} \right)^{q-2} + x, \quad (6)$$

where $a = -c_0c_1^2$, $b = -c_1$, $x_2 = -1/c_0c_1$ and $F(x_2) = -(c_0c_1^2 + 1)/(c_0c_1)$. When $a = -1$, one has $F(\delta) = -c_1$ for any $\delta \in \mathbb{F}_q^* \setminus \{x_2\}$, and $F(x_2) = -2c_1$. Therefore $V_F = \{0, -c_1, -2c_1\}$.

The cases $a = 1$ and $a \neq \pm 1$, or with the above notation $c_0c_1^2 = -1$ and $c_0c_1^2 \neq \pm 1$ yield polynomials F in (6) with $|V_F| = q-2$. By straightforward calculations one can see that if $c_0c_1^2 \neq \pm 1$, the polynomial $F(x)$ in (6) has the value set $V_F = \mathbb{F}_q \setminus \{-c_1, -1/c_0c_1\}$, and if $c_0c_1^2 = -1$, then $V_F = \mathbb{F}_q \setminus \{c, -c\}$ where $c = c_1$. □

Corollary 4. *The following polynomials have value sets of cardinalities 3 and $q-2$.*

(i) *Any polynomial $F \in \mathcal{F}_{q,2}$ of the form*

$$F(x) = \left(\left(\left(\frac{1}{c^2}x \right)^{q-2} + c \right)^{q-2} - \frac{1}{c} \right)^{q-2} + x,$$

where $c \in \mathbb{F}_q^$, has the value set $V_F = \{0, -c, -2c\}$. The value set count for F is (v_0, v_1, v_{q-2}) , where $v_0 = q-3, v_1 = 2, v_{q-2} = 1$.*

(ii) Any polynomial $F \in \mathcal{F}_{q,2}$ of the form

$$F(x) = \left(\left(\left(\frac{-1}{c^2} x \right)^{q-2} + c \right)^{q-2} - \frac{1}{c} \right)^{q-2} + x,$$

has the value set $V_F = \mathbb{F}_q \setminus \{c, -c\}$, where $c \in \mathbb{F}_q^*$ is arbitrary. The value set count in this case is (v_0, v_1, v_3) where $v_0 = 2$, $v_1 = q - 3$, and $v_3 = 1$.

Proof. Immediate from the above proof by putting $c = c_1$. \square

Theorem 5. The spectrum of the family $\mathcal{F}_{q,3}$ is

$$v(\mathcal{F}_{q,3}) = \begin{cases} \{2, 4, q-3, q-2, q\} & \text{if } q \equiv 0 \pmod{3}, \\ \{3, 4, q-3, q-2, q\} & \text{if } q \equiv 1 \pmod{3}, \\ \{3, 4, q-3, q-2\} & \text{if } q \equiv 2 \pmod{3}. \end{cases}$$

Proof. When $F \in \mathcal{F}_{q,3}$, the corresponding polynomial $f(x) = P_3(c_0, c_1, c_2, c_3; x)$ in (3) satisfies $a = c_0(c_1c_2 + 1)^2$, $b = c_2(c_1c_2 + 1)$, $c_3 = -c_1/(c_1c_2 + 1)$.

We first focus on small value sets, i.e., $|V_F| = 2, 3, 4$. As we have seen in Theorem 1, these values occur when $a = -1$ or $c_0 = -1/(c_1c_2 + 1)^2$. In this case $x_2 = (c_1c_2 + 1)^2/c_1$, $x_3 = b = c_2(c_1c_2 + 1)$ and hence we get $F(x_2) = (c_1c_2 + 1)(2c_1c_2 + 1)/c_1$, $F(x_3) = (c_1^2c_2^2 - 1)/c_1$. Since $F(\mathbb{F}_q \setminus \mathbf{O}_3) = \{x_3\}$, we have $F(\mathbf{O}_3) \cap F(\mathbb{F}_q \setminus \mathbf{O}_3) = \emptyset$.

One can easily see that $|F(\mathbf{O}_3)| \leq 3$ exactly when $c_1c_2 \in \{-2, -1/2, 1\}$. For instance when $c_1c_2 = 1$, one gets $F(x_1) = F(x_3) = 0$ and $F(x_2) = 6c_2$ while $F(\delta) = 2c_2$ for $\delta \in \mathbb{F}_q \setminus \mathbf{O}_3$. Hence for $\text{char}(\mathbb{F}_q) \neq 3$ we obtain $V_F = \{0, 2c_2, 6c_2\}$ with the value set count (v_0, v_1, v_2, v_{q-3}) , where $v_0 = q - 3$, $v_1 = v_2 = v_{q-3} = 1$.

For $c_1c_2 = -1/2$ one gets $F(x_1) = F(x_2) = 0$, $F(x_3) = 3c_2/2$, which yields $V_F = \{0, 3c_2/2, c_2/2\}$, since $F(\delta) = c_2/2$ for every $\delta \in \mathbb{F}_q \setminus \mathbf{O}_3$. Note that $|F^{-1}(0)| = 2$, $|F^{-1}(3c_2/2)| = 1$ and $|F^{-1}(c_2/2)| = q - 3$ when $\text{char}(\mathbb{F}_q) \neq 3$ and hence the value set count is (v_0, v_1, v_2, v_{q-3}) , where $v_0 = q - 3$, $v_1 = v_2 = v_{q-3} = 1$. The case $c_1c_2 = -2$ can be dealt with similarly to yield $V_F = \{0, -3c_2/2, -c_2\}$, so that $|V_F| = 2$ or 3 depending on $\text{char}(\mathbb{F}_q)$.

When $d = c_1c_2 \notin \{-2, -1/2, 0, 1\}$, one obtains polynomials F with the value set $V_F = \{0, (d+1)(2d+1)/c, (d^2-1)/c, d(d+1)/c\}$, with $c = c_1$.

For the large value sets of sizes $q-3, q-2$ and q that are obtained when $c_0 \neq -1/(c_1c_2 + 1)^2$, we consider the cases

- (i) $c_0 = -1/(c_1c_2 + 1)$, implying $F(x_1) \cap F(\mathbb{F}_q \setminus \mathbf{O}_3) = \emptyset$,
- (ii) $c_0 = 1/c_1c_2(c_1c_2 + 1)^2$, implying $F(x_2) \cap F(\mathbb{F}_q \setminus \mathbf{O}_3) = \emptyset$,

(iii) $c_0 = -(c_1c_2)/(c_1c_2 + 1)^3$, implying $F(x_3) \cap F(\mathbb{F}_q \setminus \mathbf{O}_3) = \emptyset$.

When $q \equiv 1 \pmod{3}$ or $\text{char}(\mathbb{F}_q) = 3$ the equation $c_1^2c_2^2 + c_1c_2 = -1$ has a solution and hence the equations (i),(ii),(iii) above are simultaneously satisfied. By straightforward calculations one can see in this case that $|F(\mathbf{O}_3)| = 3$, and with $F(\mathbf{O}_3) \cap F(\mathbb{F}_q \setminus \mathbf{O}_3) = \emptyset$, one obtains $|V_F| = q$.

When $q \equiv 0 \pmod{3}$ and $c_0 = 1/(c_1c_2 + 1)^3$ we have $F(x_2) = F(x_3) = F(\delta_1)$ for a unique $\delta_1 \in \mathbb{F}_q \setminus \mathbf{O}_3$. Moreover, if $(c_1c_2 + 1)^2 \neq -1$ then there exists a unique $\delta_2 \in \mathbb{F}_q \setminus \mathbf{O}_3$ with $F(x_1) = F(\delta_2)$. Therefore we get $|V_F| = q - 3$. In this case $V_F = \mathbb{F}_q \setminus \{cd(d-1), cd(-d^2-1), cd^2(-d+1)\}$ where $d = c_1c_2 + 1$ and $c = 1/c_1$ is arbitrary in \mathbb{F}_q^* . Otherwise if $(c_1c_2 + 1)^2 = -1$, that is $q \equiv 9 \pmod{12}$, then we have $|V_F| = q - 2$. In this case we have $V_F = \mathbb{F}_q \setminus \{c(-1-d), c(d-1)\}$ where $d = c_1c_2 + 1$ and $c = 1/c_1$ is arbitrary in \mathbb{F}_q^* .

When $q \equiv 1 \pmod{3}$, $c_0 = 1/(c_1c_2 + 1)^2$ and $c_1c_2 \notin \{-2, -1/2, 1\}$ then there are uniquely determined $\delta_1, \delta_2, \delta_3 \in \mathbb{F}_q \setminus \mathbf{O}_3$ with $F(x_1) = F(\delta_1)$, $F(x_2) = F(\delta_2)$ and $F(x_3) = F(\delta_3)$. Hence we get $|V_F| = q - 3$. In this case $V_F = \mathbb{F}_q \setminus \{cd(d-1), cd(-d-1), cd(-d+1)\}$ where $d = c_1c_2 + 1$ and $c = 1/c_1$ is in \mathbb{F}_q^* . On the other hand, if $c_1c_2 = 1$ then there exist uniquely determined elements $\delta_1, \delta_2 \in \mathbb{F}_q \setminus \mathbf{O}_3$ with $F(x_1) = F(\delta_1)$, $F(x_3) = F(\delta_2)$ and all other values are attained once. In this case we get $V_F = \mathbb{F}_q \setminus \{2c, -6c\}$ where $c = 1/c_1$ is in \mathbb{F}_q^* . Hence $|V_F| = q - 2$.

When $q \equiv 2 \pmod{3}$ we also get $|V_F| = q - 2$ or $q - 3$. There are various possibilities for the values $q - 2$ or $q - 3$ to be attained. For instance if $q \equiv 5 \pmod{12}$, $c_0 = -1/(c_1c_2 + 1)$ and $(c_1c_2 + 1)^2 = -1$, then there exists a unique $\delta \in \mathbb{F}_q \setminus \mathbf{O}_3$ with $F(\delta) = F(x_2) = F(x_3)$. In this case $V_F = \mathbb{F}_q \setminus \{-c - cd, -c + cd\}$ where $d = (c_1c_2 + 1)$ and $c = 1/c_1$ is in \mathbb{F}_q^* .

If, on the other hand, $q \equiv 11 \pmod{12}$, $c_0 = 1/c_1c_2(c_1c_2 + 1)$, and $c_1c_2 \notin \{0, -1, -1/2\}$, then there exist uniquely determined $\delta_1, \delta_2 \in \mathbb{F}_q \setminus \mathbf{O}_3$ such that $F(\delta_1) = F(x_1) = F(x_2) = 0$ and $F(\delta_2) = F(x_3)$. In this case $V_F = \mathbb{F}_q \setminus \{cd(d+1), -c(d+1)^2, -d^2c\}$ where $d = c_1c_2$ and $c = 1/c_1$ is in \mathbb{F}_q^* . \square

Corollary 6. *Depending on q , the following polynomials have value sets of cardinalities $q, q - 2$, and $q - 3$.*

(i) *Let $q \equiv 1 \pmod{3}$. Any polynomial $F(x)$ of the form*

$$F(x) = \left(\left(\left(\left(\frac{-x}{d+1} \right)^{q-2} + c \right)^{q-2} + \frac{d}{c} \right)^{q-2} - \frac{c}{d+1} \right)^{q-2} + x$$

is a permutation, if d satisfies $d^2 + d + 1 = 0$, in other words d is a primitive third root of unity, and $c \in \mathbb{F}_q^*$ is arbitrary. Note that $F(\delta) = f(\delta) + \delta$ for $\delta \in \mathbb{F}_q$, where f is obtained by altering $g(x) = -(d+1)x + (d^2 + d)/c$ at three elements; $x_1 = 0, x_2 = (d+1)/c, x_3 = d/c$, as in (2).

(ii) Let $q \equiv 5 \pmod{12}$. Any polynomial $F(x)$ of the form

$$F(x) = \left(\left(\left(\left(\frac{-x}{d+1} \right)^{q-2} + c \right)^{q-2} + \frac{d}{c} \right)^{q-2} - \frac{c}{d+1} \right)^{q-2} + x$$

with $(d+1)^2 = -1$, $c \in \mathbb{F}_q^*$ arbitrary, has the value set $V_F = \mathbb{F}_q \setminus \{d/c, (-d-2)/c\}$. Here $m(-1/c) = 3$, and $m(\alpha) = 1$, for all $\alpha \in V_F, \alpha \neq -1/c$. Hence the maximum value count is 3.

(iii) Let $q \equiv 11 \pmod{12}$. Any polynomial $F(x)$ of the form

$$F(x) = \left(\left(\left(\left(\frac{x}{d(d+1)} \right)^{q-2} + c \right)^{q-2} + \frac{d}{c} \right)^{q-2} - \frac{c}{d+1} \right)^{q-2} + x$$

has the value set $V_F = \mathbb{F}_q \setminus \{d(d+1)/c, -(d+1)^2/c, -d^2/c\}$, where $c \in \mathbb{F}_q^*$ is arbitrary and $d \notin \{-1, -1/2, 0\}$. Here $m((-d^2 - d - 1)c) = 2, m(0) = 3$ and $m(\alpha) = 1$ for all the other $q - 5$ elements $\alpha \in V_F$. Hence the maximum value count is 3. The value set count is (v_0, v_1, v_2, v_3) where $v_0 = 3, v_1 = q - 5, v_2 = 1, v_3 = 1$.

Proof. By putting $c_1 c_2 = d$ and $c = c_1$ the results follow immediately from the proof of the above theorem. \square

Remark 7. The polynomial $F(x)$ in (i) in the above corollary, and $F(x) - x = f(x)$ are both permutations, i.e., $f(x)$ is a complete mapping.

Corollary 8. Any polynomial $F(x)$ of the form

$$F(x) = \left(\left(\left(\left(\frac{-x}{(d+1)^2} \right)^{q-2} + c \right)^{q-2} + \frac{d}{c} \right)^{q-2} - \frac{c}{d+1} \right)^{q-2} + x$$

has the value set $V_F = \{0, (d^2 - 1)/c, (d+1)d/c, (d+1)(2d+1)/c\}$, where $c \in \mathbb{F}_q^*$ is arbitrary and $d \notin \{-2, -1/2, 0, 1\}$. The value set count is (v_0, v_1, v_{q-3}) with $v_0 = q - 4, v_1 = 3, v_{q-3} = 1$.

4 Polynomials with prescribed $|V_f|$

We first note that the polynomials $F \in \mathcal{F}_{q,n}$ can be easily described as in (3), when n is small enough and \mathbf{O}_n is given. We summarize the procedure given in [9], for the sake of completeness. Recall that $x_n = -b/a = -\beta_n/\alpha_n$. Given $g(x) = ax + b$, and \mathbf{O}_n , we can therefore use the equations (4) to calculate c_0, c_1, \dots, c_n , and hence $F(x) = P_n(c_0, \dots, c_n : x) + x$. Put $\alpha_n = \epsilon a$, $\beta_n = \epsilon b$, $\alpha_{n+1} = 0$ and $\beta_{n+1} = \epsilon$, for a variable ϵ . From (4) one gets

$$c_i = \frac{\beta_{i+1} + x_{i-1}\alpha_{i+1}}{\beta_i + x_{i-1}\alpha_i}, \quad 2 \leq i \leq n.$$

Therefore one can recursively calculate the exact values for c_n, c_{n-1}, \dots, c_3 , and values for $\alpha_{n-1}, \beta_{n-1}, \dots, \alpha_2, \beta_2$ as multiples of ϵ . In the final step, using $\beta_0 = 1$, $\beta_1 = 0$, and hence $\beta_2 = 1$, one obtains the value for ϵ . Then one can find $c_2 = \beta_3$, $\alpha_1 = \alpha_3 - c_2\alpha_2$, $c_1 = \alpha_2/\alpha_1$ and $c_0 = \alpha_1$.

The following theorem gives examples of polynomials F with small and large value sets, depending on q, n .

Theorem 9. *The following choices of $g(x)$ and \mathbf{O}_n yield polynomials $F \in \mathcal{F}_{q,n}$ with $|V_F| = n + 1$, $|V_F| = q - n$, $|V_F| \geq q - n$, where the maximum count for V_F is two, and $|V_F| = 2, 3$.*

- (i) *Let $q = p^r$, $p > n(n + 1)$. Then the polynomial $F(x)$ as in (5), with $a = -1$, $b = n(n - 1)/2$ and $\mathbf{O}_n = \{x_i : x_1 = 0, x_i = x_{i-1} + i - 1, 2 \leq i \leq n\}$ has $|V_F| = n + 1$. The value set count for F is (v_0, v_1, v_{q-n}) , where $v_0 = q - n - 1$, $v_1 = n$, $v_{q-n} = 1$.*
- (ii) *Let $q \equiv 1 \pmod n$, and $a \in \mathbb{F}_q^*$, with $\text{ord}(-a) = n$. Then for any $b \in \mathbb{F}_q^*$ and $\mathbf{O}_n = \{x_i : x_i = b \sum_{j=0}^{n-i} (-1/a)^{j+1}, 1 \leq i \leq n\}$, the polynomial $F(x)$ as in (5) has $|V_F| = q - n$. The value set count for F is (v_0, v_1, v_{n+1}) , where $v_0 = n$, $v_1 = q - n - 1$, $v_{n+1} = 1$.*
- (iii) *Let $q \equiv 1 \pmod{2n}$, $a, b \in \mathbb{F}_q^*$ with $\text{ord}(a) = 2n$. Put $z_i = ba^{i-1}$, and $x_i = (z_i - b)/(a + 1)$ for $1 \leq i \leq n$. Then the polynomial $F(x)$ as in (5) has $|V_F| \geq q - n$, and the maximum count for V_F is at most 2.*
- (iv) *Let $q = p^r$, $p > n - 2$. Then the polynomial $F(x)$ as in (5), with $a = -1$, $b \in \mathbb{F}_q^*$ and $\mathbf{O}_n = \{x_i : x_i = (1 - i)b, 1 \leq i \leq n - 1, x_n = b\}$ has $V_F = \{0, b, nb\}$, with $m(0) = n - 1$, $m(b) = q - n$, $m(nb) = 1$.*

Proof. (i) Note that $x_n = b = n(n-1)/2$. Since $a = -1$, we have, as in Theorem 1, $F(\delta) = b$ for every $\delta \in \mathbb{F}_q \setminus \mathbf{O}_n$. The choice of \mathbf{O}_n gives $|F(\mathbf{O}_n)| = n$ and $b \notin F(\mathbf{O}_n)$. Indeed $V_F = \{0\} \cup \{b+i : 0 \leq i \leq n-1\}$. The value set count also follows trivially.

(ii) We show that by this choice of \mathbf{O}_n we get $F(\mathbf{O}_n) = \{0\}$ and $F(\mathbf{O}_n) \cap F(\mathbb{F}_q \setminus \mathbf{O}_n) = \{0\}$. Firstly note that $x_n = -b/a$, and $\text{ord}(-1/a) = n$ implies $x_1 = 0$, $|\mathbf{O}_n| = n$, as required. Recall that $F(x_i) = ax_{i-1} + x_i + b$ for $2 \leq i \leq n$. Hence

$$F(x_i) = a((-b/a) \sum_{j=0}^{n-i+1} (-1/a)^j) + ((-b/a) \sum_{j=0}^{n-i} (-1/a)^j) + b.$$

By straightforward calculations one gets $F(x_i) = 0$ for every $1 \leq i \leq n$. On the other hand there exists $\delta = -b/(a+1) \in \mathbb{F}_q^*$ with $F(\delta) = 0$. Note that $\delta \notin \mathbf{O}_n$, since $-b/(a+1) = (-b/a) \sum_{j=0}^{n-i} (-1/a)^j$ for some $1 \leq i \leq n$ yields a contradiction. Recalling that F is linear on $\mathbb{F}_q \setminus \mathbf{O}_n$, we obtain $|V_F| = q - n$. Note that $m(0) = n+1$, and $m(\alpha) = 1$ for all the other elements $\alpha \in V_F \setminus \{0\}$.

(iii) By this choice of \mathbf{O}_n , it is obvious that $F(x_i) \neq F(x_j)$ for $1 \leq i \neq j \leq n$. Therefore the only elements in V_F with multiplicity ≥ 2 can belong to the set $F(\mathbf{O}_n) \cap F(\mathbb{F}_q \setminus \mathbf{O}_n)$. However F is a permutation on $\mathbb{F}_q \setminus \mathbf{O}_n$ and hence no element can have multiplicity exceeding 2.

(iv) Clearly $F(x_i) = 0$ for $1 \leq i \leq n-1$ and $F(x_n) = nx_n = nb$. The condition $a = -1$ implies $F(\mathbb{F}_q \setminus \mathbf{O}_n) = \{x_n\} = \{b\}$. Therefore $|V_F| = 2$ or 3, depending on the characteristic p .

□

Remark 10. The polynomials in Theorem 9, parts (ii) and (iii) are evenly distributed in the following sense. When $F(x)$ is as in part (ii), the multiplicity of any non-zero element in V_F is one. When $F(x)$ is as in part (iii), the maximum count for V_F is two.

The following result is a corollary of Theorem 9. We state it as a theorem since it may be of independent interest.

Theorem 11. Let U be a subgroup of the multiplicative group \mathbb{F}_q^* . Suppose that $|U| = n$, $U = \langle \alpha \rangle$, and $c \in \mathbb{F}_q^*$, $c \neq 1$. Then the polynomial $F \in \mathcal{F}_{q,n}$, defined as in (5), with $a = -1/\alpha$ and $b = -ac$ has $V_F = \mathbb{F}_q \setminus cU$.

Proof. For $a = -1/\alpha$ and $b = -ac$, choose \mathbf{O}_n as in the proof of Theorem 9(ii). Then the missing values in V_F are exactly those, given by $G(x_i) = (a+1)x_i + b = (a+1)((-b/a) \sum_{j=0}^{n-i} (-1/a)^j) + b = (-b/a)(-1/a)^{n-i}$ for $1 \leq i \leq n$. \square

We end this note by an example, illustrating the procedure we explained in the beginning of this section.

Example 12. Let $p = 13$, $n = 4$ and $g(x) = -x+2$. Then $x_4 = 2$. Suppose that $x_i = (1-i)x_4$ for $i = 2, 3$, as in Theorem 9, part (iv). Hence $V_F = \{0, 2, 8\}$. To describe $F(x)$, we put $\alpha_4 = -\epsilon$, $\beta_4 = 2\epsilon$, $\beta_5 = \epsilon$. Recall that $\alpha_5 = 0$. Then $c_4 = 11$ and $\alpha_3 = \alpha_5 - c_4\alpha_4 = -2\epsilon$, $\beta_3 = \beta_5 - c_4\beta_4 = 5\epsilon$. We obtain recursively $c_3 = (\beta_4 + x_2\alpha_4)/(\beta_3 + x_2\alpha_3) = 12$, $\alpha_2 = \alpha_4 - c_3\alpha_3 = 10\epsilon$, $\beta_2 = \beta_4 - c_3\beta_3 = 7\epsilon = 1$. Therefore $\epsilon = 2$, $c_2 = \beta_3 = 10$, $\alpha_1 = \alpha_3 - c_2\alpha_2 = 4$, $c_1 = \alpha_2/\alpha_1 = 5$ and $c_0 = \alpha_1 = 4$, giving

$$F(x) = \left(\left(\left(\left((4x)^{11} + 5 \right)^{11} + 10 \right)^{11} + 12 \right)^{11} + 11 \right)^{11} + x.$$

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